PUTNAM PRACTICE SET 30

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Problem 1. Let $f : [0,1]^2 \longrightarrow \mathbb{R}$ be a continuous function on the unit square such that the partial derivatives df/dx and df/dy exists and are continuous on the interior $(0,1)^2$. Prove or disprove whether there always exists some point $(x_0, y_0) \in (0,1)^2$ such that:

$$\frac{\mathrm{df}}{\mathrm{dx}}(x_0,y_0) = \int_0^1 f(1,y)\mathrm{dy} - \int_0^1 f(0,y)\mathrm{dy} \text{ and } \frac{\mathrm{df}}{\mathrm{dy}}(x_0,y_0) = \int_0^1 f(x,1)\mathrm{dx} - \int_0^1 f(x,0)\mathrm{dx}$$

Problem 2. Show that every positive rational number can be written as a quotient of factorials of primes (not necessarily distinct); for example,

$$\frac{6}{7} = \frac{3! \cdot 3! \cdot 5!}{7!}.$$

Problem 3. A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c, can one travel from 0 to 1 in a finite number of jumps with total cost equal to c?

Problem 4. Say that a polynomial $P \in \mathbb{R}[x,y]$ is balanced if the average value of the polynomial on each circle centered at the origin is 0, i.e.,

$$\int_C P(x,y) = 0$$

for any circle C in the cartesian plane. The balanced polynomials of degree 2021 form an $\mathbb{R}$-vector space V; find $\dim_{\mathbb{R}} V$.